Shattering-extremal set systems of small VC-dimension

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Abstract

We say that a set system $\mathcal{F} \subseteq 2^{[n]}$ shatters a given set $S \subseteq [n]$ if $2^S = \{F \cap S : F \in \mathcal{F}\}$. The Sauer inequality states that in general, a set system $\mathcal{F}$ shatters at least $|\mathcal{F}|$ sets. Here we concentrate on the case of equality. A set system is called shattering-extremal if it shatters exactly $|\mathcal{F}|$ sets. We characterize shattering extremal set systems of Vapnik-Chervonenkis dimension 1 in terms of their inclusion graphs. Also from the perspective of extremality, we relate set systems of bounded Vapnik-Chervonenkis dimension to their projections.

1 Introduction

Throughout this paper n will be a positive integer, the set $\{1, 2, \ldots, n\}$ will be referred to shortly as $[n]$, the power set of it as $2^{[n]}$ and the family of subsets of size k as $\binom{[n]}{k}$.

The central notion of our study is shattering. We say that a set system $\mathcal{F} \subseteq 2^{[n]}$ shatters a given set $S \subseteq [n]$ if $2^S = \{F \cap S : F \in \mathcal{F}\}$. The family of subsets of $[n]$ shattered by $\mathcal{F}$ is denoted by $Sh(\mathcal{F})$. The following inequality gives a bound on the size of $Sh(\mathcal{F})$.

Proposition 1.1 $|Sh(\mathcal{F})| \geq |\mathcal{F}|$.

The statement was proved by several authors, Aharoni and Holzman [2], Pajor [15], Sauer [18], Shelah [19]. Often it is referred to as Sauer inequality. One of the most interesting cases is the case of equality, i.e. when the set system $\mathcal{F}$ shatters exactly $|\mathcal{F}|$ sets. We call such set systems shattering extremal or s-extremal for short. Many interesting results have been obtained in connection with these combinatorial objects, among others in [4], [5], [6], [12], and [13].

The Vapnik-Chervonenkis dimension of a set system $\mathcal{F} \subseteq 2^{[n]}$, denoted by $VC - dim(\mathcal{F})$, is the maximum cardinality of a set shattered by $\mathcal{F}$. The general task of giving a good description of s-extremal systems seems to be too complex at this point. We restrict therefore our attention to the simplest cases, where the VC-dimension of $\mathcal{F}$ is bounded by some fixed natural number t.

After the introduction, in Section 2 we first investigate s-extremal set systems of VC-dimension at most 1 from a graph theoretical point of view. We give a bijection between the family of such set systems on the ground set $[n]$ and trees on $n + 1$ vertices. As a consequence one can exactly determine the number of such s-extremal set systems. In combinatorics when considering set systems with a given property it is a common step to first consider families of some special structure. According to [14] uniform set systems can't be s-extremal. As a next possibility set systems from two consecutive layers turn up. In Section 3 we prove that they are just special cases of the previous ones. After this in Section 4 we switch to an algebraic point of view and investigate bases the polynomial ideals attached to extremal set systems. The main result of Section 5 is a connection between s-extremal set systems of VC-dimension t and their projections. At the end we propose an open problem and make some concluding remarks.

In the paper we will use the terminology of [11] for graph theoretical notions.

2   s-extremal set systems of VC-dimension at most 1

Let $\mathcal{F} \subseteq 2^{[n]}$ be an s-extremal family. Let $G_{\mathcal{F}}$ be the labelled Hasse diagram of $\mathcal{F}$ considered as a graph, i.e. a graph whose vertices are the elements of $\mathcal{F}$ and there is a directed edge going from G to F, labelled with $j \in [n]$ exactly when $F = G \cup \{j\}$. $G_{\mathcal{F}}$ will be called the inclusion graph of $\mathcal{F}$. When representing the elements of $\mathcal{F}$ by their characteristic vectors, $G_{\mathcal{F}}$ can also be considered as the subgraph in the Hamming graph $H_n = \{0, 1\}^n$ spanned by the elements corresponding to the sets in $\mathcal{F}$ with edges directed and labelled

in a natural way. Actually for the next proposition we can forget about the directions of the edges, and consider $G_{\mathcal{F}}$ as an undirected edge-labelled graph. We further assume that $VC - dim(\mathcal{F}) \leq 1$. Our aim is to characterize these kinds of s-extremal set systems in terms of their inclusion graph.

Proposition 2.1 *A set system $\mathcal{F} \subseteq 2^{[n]}$ is s-extremal and of $VC-dimension$ at most 1 iff $G_{\mathcal{F}}$ is a tree and all labels on the edges are different.*

Proof: For the 'only if' direction suppose that $\mathcal{F}$ is s-extremal and $VC - dim(\mathcal{F}) \leq 1$. According to [14], Theorem 5 we know that $G_{\mathcal{F}}$ must be isometrically embedded into H_n (i.e. for any two elements $G, F \in \mathcal{F}$ the distance between G and F is the same in $G_{\mathcal{F}}$ and in H_n). This means in particular, that $G_{\mathcal{F}}$ is connected. Next we prove that all labels on the edges of $G_{\mathcal{F}}$ are different. Suppose for contradiction, that there are two edges with the same label. W.l.o.g. we may assume that this label is 1. Since there are no two edges going out from a set with the same label, there are sets $A, B, C, D \in \mathcal{F}$, all different, such that $1 \in A \cap B$, $C = A \backslash \{1\}$ and $D = B \backslash \{1\}$. Since $A \neq B$, $A \triangle B(= (A\backslash B) \cup (B\backslash A))$ is nonempty, so there is an element $1 \neq a \in A \triangle B$. W.l.o.g. we may assume that $a \in A \backslash B$. Now

$$\{1,a\} \cap A = \{1,a\} \quad \{1,a\} \cap B = \{1\} \quad \{1,a\} \cap C = \{a\} \quad \{1,a\} \cap D = \emptyset \ .$$

So $\{1,a\}$ is shattered by $\{A, B, C, D\}$, consequently $\{1,a\} \in Sh(\mathcal{F})$, contradicting the assumption $VC - dim(\mathcal{F}) \leq 1$.

To finish with this direction note that the fact that all labels are different implies that $G_{\mathcal{F}}$ is acyclic. Suppose for contradiction that it is not the case, and $G_{\mathcal{F}}$ contains a cycle. Pick one edge from this cycle and let a be its label. On the remaining part of the cycle there must be another edge labelled with a, since it connects a set containing a with one not containing a. However this is impossible, since all labels are different. Adding the connectedness of $G_{\mathcal{F}}$, we obtain that it is actually a tree as wanted.

For the reverse direction suppose that $G_{\mathcal{F}}$ is a tree and all labels on the edges are different. It is easily seen that this implies that $G_{\mathcal{F}}$ is isometrically embedded into H_n. (Otherwise a path from a set A to B in $G_{\mathcal{F}}$ which is not a shortest in H_n would contain 2 edges with the same label, corresponding to the addition and deletion of the same element of $[n]$.)

Now we prove that $VC - dim(\mathcal{F}) \leq 1$. Suppose the contrary, namely that $\mathcal{F}$ shatters a set of size 2, e.g. $\{1,2\}$. This means that there are sets $A, B, C, D \in \mathcal{F}$ such that

$$\{1,2\} \cap A = \{1,2\} \quad \{1,2\} \cap B = \{1\} \quad \{1,2\} \cap C = \{2\} \quad \{1,2\} \cap D = \emptyset \ .$$

Consider a shortest path in $G_{\mathcal{F}}$ from A to B. Since $2 \in A \backslash B$, this shortest path has to contain an edge labelled with 2. Repeating this argument for C and D one gets another, different (since on a shortest path between A and B every set contains the element 1, on the other hand on a shortest path between C and D none of the sets does) edge with label 2, what contradicts the assumption that all labels are different.

Now we calculate $Sh(\mathcal{F})$. If $i \in [n]$ is not an edge label, then either all sets $F \in \mathcal{F}$ contain i, ore none of them does. In particular $\{i\}$ is not shattered by $\mathcal{F}$. Thus $Sh(\mathcal{F})$ consists of $\emptyset$ and the sets $\{i\}$, where i is an edge label. However all edge labels are different, so we get that $|Sh(\mathcal{F})| = |E(G_{\mathcal{F}})| + 1 = |\mathcal{F}|$ (since $G_{\mathcal{F}}$ is a tree), i.e. $\mathcal{F}$ is s-extremal. ∎

Let $\mathcal{F} \subseteq 2^{[n]}$ be an s-extremal family such that $supp(\mathcal{F}) = \cup_{F \in \mathcal{F}} F = [n]$ and $\cap_{F \in \mathcal{F}} F = \emptyset$. By Proposition 2.1 to every s-extremal family of VC-dimension at most 1 one can associate a directed edge-labelled tree $G_{\mathcal{F}}$, all edges having distinct labels. We have seen that $Sh(\mathcal{F})$ consists of $\emptyset$ and the sets $\{i\}$, where i is an edge label. On the other hand, since $\cap_{F \in \mathcal{F}} F = \emptyset$, we also have that $Sh(\mathcal{F}) = \{\emptyset\} \cup \{\{j\} \mid j \in supp(\mathcal{F}) = [n]\}$. As a consequence the tree must have n edges and thus $n+1$ vertices, i.e. such an s-extremal family has $n+1$ elements.

Now conversely suppose that we are given a directed edge-labelled tree T on $n+1$ vertices with n edges, all having different labels. This tree at the same time also defines a set system $\mathcal{F}_T = \{F_v \mid v \in T\}$. Take the edges one by one. When considering an edge with label s going from u to v, then for all vertices w closer to v than to u in the undirected tree put s into F_w. Clearly $T = G_{\mathcal{F}}$ and by the previous proposition $\mathcal{F}$ must be s-extremal.

To illustrate this, consider the following example with $n = 5$.
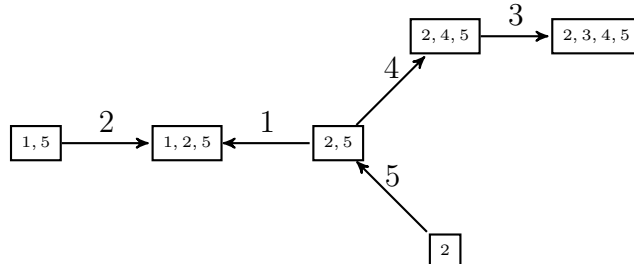
We have $\mathcal{F}_T = \{\{1, 5\}, \{1, 2, 5\}, \{2, 5\}, \{2, 4, 5\}, \{2, 3, 4, 5\}, \{2\}\}$.

This gives a bijection between the set of all s-extremal families of VC-dimension at most 1 and directed edge-labelled trees.

Theorem 2.1 *Let $n \geq 1$ be an integer. There is a one-to-one correspondence between s-extremal families $\mathcal{F} \subseteq 2^{[n]}$ of Vapnik-Chervonenkis dimension 1 with*

$supp(\mathcal{F}) = [n]$, $\cap_{F \in \mathcal{F}} F = \emptyset$ and directed edge-labelled trees on $n + 1$ vertices, all edges having a different label from $[n]$. ∎

As a corollary one can prove the following statement.

Corollary 2.1.1 *There are $2^n(n + 1)^{n-2}$ different s-extremal families $\mathcal{F} \subseteq 2^{[n]}$ of Vapnik-Chervonenkis dimension at most 1 with $supp(\mathcal{F}) = [n]$ and $\cap_{F \in \mathcal{F}} F = \emptyset$.*

Proof: There are $(n + 1)^{n-2}$ different edge labelled undirected trees on $n + 1$ vertices (see [16]), all edges having a different label from $[n]$ and each of these trees can be directed in 2^n ways. ∎

Simple examples of s-extremal set systems are down-sets, i.e. set systems $\mathcal{F}$ such that for all $i \in [n]$ $i \in F \in \mathcal{F}$ implies $F \backslash \{i\} \in \mathcal{F}$. For down-sets $Sh(\mathcal{F}) = \mathcal{F}$, so they are obviously s-extremal. One can obtain other examples from down-sets using different set system operations, e.g. bit flips. For $i \in [n]$ let φ_i be the the the ith bit flip, i.e. for $F \in 2^{[n]}$

$$\varphi_i(F) = \begin{cases} F \backslash \{i\} & \text{if } i \in F \\ F \cup \{i\} & \text{if } i \notin F. \end{cases}$$

For $\mathcal{F} \subseteq 2^{[n]}$ set $\varphi_i(\mathcal{F}) = \{\varphi_i(F) \mid F \in \mathcal{F}\}$. It is easily seen that s-extremality is invariant with respect to this operation since it keeps the family of shattered sets. However not all s-extremal set systems can be obtained in this way. For this note that in terms of the inclusion graph a bit flip in the ith coordinate corresponds just to reversing the direction of the edge with label i in $G_{\mathcal{F}}$, i.e. bit flips preserve the undirected structure of the inclusion graph. Using this we can obtain many s-extremal examples not coming from down-sets using bit flips. It is enough to pick a tree that is not a star and consider a set system corresponding to any possible orientation.

3 s-extremal set systems from two consecutive layers

For an uniform family $\mathcal{F}$ the graph $G_{\mathcal{F}}$ is not connected, hence $\mathcal{F}$ cannot be s-extremal. As a relaxation of uniformity we consider families which belong to two consecutive layers of $2^{[n]}$. The next proposition shows that extremal families among them are actually special cases of the previously studied one.

Proposition 3.1 *Let $\mathcal{F} \subseteq \binom{[n]}{k} \cup \binom{[n]}{k-1}$, $n \geq k \geq 1$ be an s-extremal family of subsets of $[n]$. Then we have $VC - dim(\mathcal{F}) \leq 1$.*

Proof: For $n = 2$ the statement can be verified by an easy case analysis. For $n > 2$ we can do induction on k. For $k = 1$ the statement is just trivial. Now suppose that $k > 1$ and the result holds for all values smaller than k. We prove that such an s-extremal family cannot shatter a subset of size 2. Suppose the contrary, namely that $\mathcal{F}$ shatters for example $\{1, 2\}$. Let

$$\mathcal{F}_0^{(n)} = \{F \mid F \in \mathcal{F} \text{ and } n \notin F\} \subseteq \binom{[n-1]}{k} \cup \binom{[n-1]}{k-1}$$

and

$$\mathcal{F}_1^{(n)} = \{F \backslash \{n\} \mid F \in \mathcal{F} \text{ and } n \in F\} \subseteq \binom{[n-1]}{k-1} \cup \binom{[n-1]}{k-2}.$$

Since $\mathcal{F}$ is s-extremal both $\mathcal{F}_0^{(n)}$ and $\mathcal{F}_1^{(n)}$ must be s-extremal (it follows easily from the proof of the Sauer inequality, see e.g. [4]) and for the shattered sets we have that

$$Sh(\mathcal{F}) = Sh(\mathcal{F}_0^{(n)}) \cup Sh(\mathcal{F}_1^{(n)}) \cup \{F \cup \{n\} \mid F \in Sh(\mathcal{F}_0^{(n)}) \cap Sh(\mathcal{F}_1^{(n)})\}.$$

Since $n > 2$, by the induction hypothesis $\{1, 2\} \in Sh(\mathcal{F}_1^{(n)})$ cannot hold, thus we have $\{1, 2\} \in Sh(\mathcal{F}_0^{(n)})$. In this way we constructed an s-extremal family with the same properties but on a smaller ground set. Continuing this we get to an s-extremal family $\mathcal{F} \subseteq \binom{[k]}{k} \cup \binom{[k]}{k-1}$ that shatters $\{1, 2\}$. However this is easily seen to be impossible, because for any $F \in \mathcal{F}$ we have $|F \cap \{1, 2\}| \geq 1$. This finishes the proof. ∎

Using essentially the same argument one can prove the following:

Proposition 3.2 *Let $\mathcal{F} \subseteq \binom{[n]}{k} \cup \binom{[n]}{k-1} \cup \cdots \cup \binom{[n]}{k-t+1}$, $n \geq k \geq t - 1 \geq 1$ be an s-extremal family of subsets of $[n]$. Then we have $VC-dim(\mathcal{F}) \leq t-1$.* ∎

We return now to the situation when $\mathcal{F} \subseteq \binom{[n]}{k} \cup \binom{[n]}{k-1}$ for some $n \geq k \geq 1$ and $supp(\mathcal{F}) = [n]$, $\cap_{F \in \mathcal{F}} F = \emptyset$. Proposition 2.1 says in this case that $\mathcal{F}$ is s-extremal iff $G_{\mathcal{F}}$ (the undirected version) is a tree and all labels on the edges are different. As before, we also have that this tree has $n + 1$ vertices and n edges. Permuting the labels on the edges corresponds just to a permutation of the ground set, so if we want to characterize s-extremal set systems up to isomorphism we can freely omit the labels from the edges.

Now suppose that we are given a tree T on $n+1$ vertices having n edges. T can also be viewed as a bipartite graph (since it is acyclic, and so contains no odd cycles) with partition classes $\mathcal{A}, \mathcal{B}$. Direct all edges from $\mathcal{A}$ to $\mathcal{B}$, and let $\mathcal{F}$ be as before the set system this directed tree just defines. It is easily seen

that we have $\mathcal{F} \subseteq \binom{[n]}{k} \cup \binom{[n]}{k-1}$, where $k = |\mathcal{A}|$ and using the characterization of s-extremal families we also get that $\mathcal{F}$ is s-extremal. If we swap the role of $\mathcal{A}$ and $\mathcal{B}$ we get the "dual" set system

$$\mathcal{F}' = \{[n] \backslash F \mid F \in \mathcal{F}\} \subseteq \binom{[n]}{n-k+1} \cup \binom{[n]}{n-k},$$

which is clearly also s-extremal using the same reasoning.

Summarizing the preceding discussion, we have the following:

Theorem 3.1 *Up to isomorphism and the operation of taking the "dual" of a set system, there is a one to one correspondence between s-extremal set systems $\mathcal{F}$ from two consecutive layers on the ground set $[n]$ (supp($\mathcal{F}$) = $[n]$ and $\cap_{F \in \mathcal{F}} F = \emptyset$) and trees on $n+1$ vertices. The bijection is realized via the map $\mathcal{F} \rightarrow G_{\mathcal{F}}$.* ∎

4 Ideal bases of s-extremal set systems of VC-dimension at most 1

Take a family $\mathcal{F} \subseteq 2^{[n]}$ and let $\mathbb{F}$ be a field. For a set $F \subseteq [n]$ let $v_F \in \{0,1\}^n$ be its characteristic vector, i.e. the the ith coordinate of v_F is 1 exactly when $i \in F$. One can associate to $\mathcal{F}$ a polynomial ideal $I(\mathcal{F}) \lhd \mathbb{F}[x_1, \ldots, x_n]$, the vanishing ideal of the set of characteristic vectors of the elements of $\mathcal{F}$:

$$I(\mathcal{F}) = \{f(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n] \mid f(v_F) = 0 \text{ for all } F \in \mathcal{F}\}.$$

$I(\mathcal{F})$ carries a lot of information about the set system. For this connection among $\mathcal{F}$ and $I(\mathcal{F})$ see [17] and [4].

If one is working with polynomial ideals it is advantageous to have a good ideal basis. Now we briefly introduce one such class of bases, namely Gröbner bases. For details we refer to [7], [8], [9], [10], and [1].

A total order $\prec$ on the monomials composed from variables $x_1, x_2, \ldots, x_n$ is a term order, if 1 is the minimal element of $\prec$, and $\prec$ is compatible with multiplication with monomials. One important term order is the lexicographic (lex for short) order. We have $x_1^{w_1} \ldots x_n^{w_n} \prec_{\text{lex}} x_1^{u_1} \ldots x_n^{u_n}$ if and only if $w_i < u_i$ holds for the smallest index i such that $w_i \neq u_i$. Reordering the variables gives an other lex term order.

The leading monomial $lm(f)$ of a nonzero polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$ is the largest monomial (with respect to $\prec$) which appears with nonzero coefficient in f, when written as the usual linear combination of monomials. We denote the set of all leading monomials of polynomials of a given ideal

$I \lhd \mathbb{F}[x_1, \ldots, x_n]$ by $Lm(I) = \{lm(f) : f \in I\}$, and we simply call them the leading monomials of I. A monomial is called a standard monomial of I, if it is not a leading monomial of any $f \in I$. Let $Sm(I)$ denote the set of standard monomials of I. Obviously, a divisor of a standard monomial is again in $Sm(I)$. Standard monomials have some very nice properties, among other things they form a linear basis of the $\mathbb{F}$-vector space $\mathbb{F}[x_1, \ldots, x_n]/I$.

A finite subset $\mathbb{G} \subseteq I$ is a Gröbner basis of I, if for every $f \in I$ there exists a $g \in \mathbb{G}$ such that $lm(g)$ divides $lm(f)$. It is not hard to verify that $\mathbb{G}$ is actually a basis of I, that is, $\mathbb{G}$ generates I as an ideal of $\mathbb{F}[x_1, \ldots, x_n]$. It is a fundamental fact that every nonzero ideal I of $\mathbb{F}[x_1, \ldots, x_n]$ has a Gröbner basis ([1] Corollary 1.6.5).

Gröbner bases and standard monomials turned out to be very useful when studying s-extremal set systems. Let $\mathcal{F} \subseteq 2^{[n]}$ be a set system and $I(\mathcal{F})$ its vanishing ideal. Subsets of $[n]$ can be identified with square-free monomials via the map $H \to \prod_{i \in H} x_i$. With this identification in mind one can prove that $Sh(\mathcal{F})$, viewed as a set of monomials, is just the union of the sets of standard monomials of $I(\mathcal{F})$ for all lexicographic term orders.

$$Sh(\mathcal{F}) = \bigcup_{lex\ term\ orders} Sm(I(\mathcal{F}))$$

On the other hand for vanishing ideals we have that $|Sm(I(\mathcal{F}))| = |\mathcal{F}|$ for all term orders. These facts altogether result that a set system $\mathcal{F}$ is s-extremal iff the set of standard monomials is the same for all lexicographic term orders. This algebraic characterization of s-extremal set systems leads to an efficient algorithm for testing s-extremality of a set system and offers also the possibility to generalize the notion to arbitrary sets of vectors. (For more details and proofs see [17]).

As an application of Proposition 2.1, we determine the Gröbner bases of s-extremal set systems of VC-dimension 1. Suppose that we are given a family $\mathcal{F} \subseteq 2^{[n]}$ together with $Sh(\mathcal{F})$. According to Subsection 4.2. of [17] one can construct a Gröbner basis of $I(\mathcal{F})$ as follows. For a pair of sets $H \subseteq S \subseteq [n]$ define the following polynomial
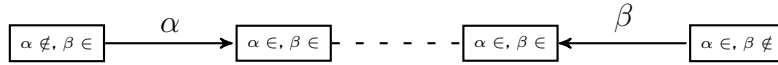
$$f_{S,H} = (\prod_{j \in H} x_j)(\prod_{i \in S \setminus H} (x_i - 1)).$$

Now if $S \notin Sh(\mathcal{F})$, then there exists a set $H \subseteq S$ such that there is no set $F \in \mathcal{F}$ with $F \cap S = H$. For this set H we have $f_{S,H} \in I(\mathcal{F})$. If the set S is minimal (i.e. all proper subsets S' of S are in $Sh(\mathcal{F})$) and $\mathcal{F}$ is s-extremal, then we also have uniqueness for the corresponding H. Moreover in the s-extremal case the collection of all these $f_{S,H}$ polynomials corresponding to minimal elements outside $Sh(\mathcal{F})$ together with $\{x_i^2 - x_i, i \in [n]\}$ form a Gröbner basis of $I(\mathcal{F})$ for all term orders. Actually more is true:

Proposition 4.1 *([17])* $\mathcal{F} \subseteq 2^{[n]}$ *is s-extremal iff there are polynomials of the form* $f_{S,H}$*, which together with* $\{x_i^2 - x_i, i \in [n]\}$ *form a Gröbner basis of* $I(\mathcal{F})$ *for all term orders.* ∎
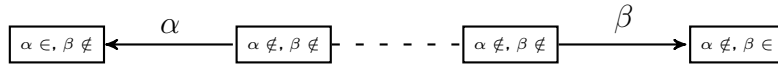
If we restrict ourselves to s-extremal set systems of VC-dimension 1, things become very simple. Suppose that $\mathcal{F} \subseteq 2^{[n]}$ is a set system such that $VC-dim(\mathcal{F}) = 1$, $supp(\mathcal{F}) = [n]$ and $\cap_{F \in \mathcal{F}} F = \emptyset$. In this case $Sh(\mathcal{F})$ is the collection of all sets of size at most 1, so the minimal sets outside $Sh(\mathcal{F})$ are exactly the sets of size 2. Fix one such set $S = \{\alpha, \beta\}$, and consider the 2 edges in the inclusion graph $G_{\mathcal{F}}$ labelled by α and β. From Proposition 2.1 we know that $G_{\mathcal{F}}$ is a tree. Consider the unique path connecting the 2 edges. There are 4 possibilities:

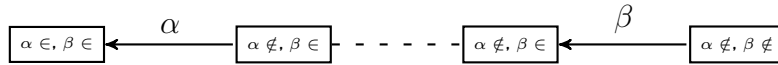- The edges are directed towards each other on this path:

$$\boxed{\alpha \notin, \beta \in} \xrightarrow{\alpha} \boxed{\alpha \in, \beta \in} \;\text{-- -- --}\; \boxed{\alpha \in, \beta \in} \xleftarrow{\beta} \boxed{\alpha \in, \beta \notin}$$

  In this case the corresponding set H is $\emptyset$, so $f_{S,H} = (x_\alpha - 1)(x_\beta - 1)$. Indeed then every $F \in \mathcal{F}$ contains either α or β.

- The edges are directed away from each other each other on this path:

$$\boxed{\alpha \in, \beta \notin} \xleftarrow{\alpha} \boxed{\alpha \notin, \beta \notin} \;\text{-- -- --}\; \boxed{\alpha \notin, \beta \notin} \xrightarrow{\beta} \boxed{\alpha \notin, \beta \in}$$
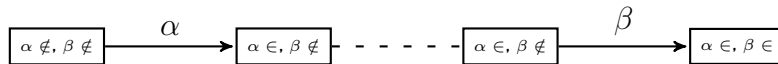
  In this case the corresponding set H is $\{\alpha, \beta\}$, so $f_{S,H} = x_\alpha x_\beta$. No $F \in \mathcal{F}$ contains $\{\alpha, \beta\}$.

- The edges are directed in the same direction towards the edge with label α on this path:

$$\boxed{\alpha \in, \beta \in} \xleftarrow{\alpha} \boxed{\alpha \notin, \beta \in} \;\text{-- -- --}\; \boxed{\alpha \notin, \beta \in} \xleftarrow{\beta} \boxed{\alpha \notin, \beta \notin}$$

  In this case the corresponding set H is $\{\alpha\}$, so $f_{S,H} = x_\alpha(x_\beta - 1)$. If $\alpha \in F$ for some $F \in \mathcal{F}$ then $\beta \in F$ as well.

- The edges are directed in the same direction towards the edge with label β on this path:

$$\boxed{\alpha \notin, \beta \notin} \xrightarrow{\alpha} \boxed{\alpha \in, \beta \notin} \;\text{-- -- --}\; \boxed{\alpha \in, \beta \notin} \xrightarrow{\beta} \boxed{\alpha \in, \beta \in}$$

  Similarly to the previous case $H = \{\beta\}$, so $f_{S,H} = (x_\alpha - 1)x_\beta$.

Now if we have $G_{\mathcal{F}}$, then using the above case analysis, one can easily compute a Gröbner basis for $I(\mathcal{F})$. If we want just a basis of $I(\mathcal{F})$ and not necessarily a Gröbner basis, we do not need to consider all pairs. Consider 3 consecutive edges in $G_{\mathcal{F}}$, i.e. a path of length 3 with labels α, β, γ.
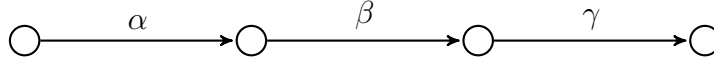
They define 3 pairs and hence 3 polynomials, $f_{\alpha,\beta} = (x_\alpha - \varepsilon_\alpha)(x_\beta - \varepsilon_\beta)$, $f_{\alpha,\gamma} = (x_\alpha - \varepsilon_\alpha)(x_\gamma - \varepsilon_\gamma)$, $f_{\beta,\gamma} = (x_\beta - 1 + \varepsilon_\beta)(x_\gamma - \varepsilon_\gamma)$, where ε_α, ε_β and ε_γ are 0 or 1 depending on the orientations of the edges. However

$$(x_\gamma - \varepsilon_\gamma)f_{\alpha,\beta} - (x_\alpha - \varepsilon_\alpha)f_{\beta,\gamma} = (1 - 2\varepsilon_\beta)f_{\alpha,\gamma},$$

where $1 - 2\varepsilon_\beta$ is either 1 or -1, so $f_{\alpha,\gamma}$ is superfluous, since it can be obtained from $f_{\alpha,\beta}$ and $f_{\beta,\gamma}$. This means that when constructing a basis of $I(\mathcal{F})$ it is enough to consider only adjacent pairs of edges in $G_\mathcal{F}$.

5 s-extremal set systems of bounded VC-dimension

The ideas from the previous sections can also be used to step a bit further, and obtain results for s-extremal set systems of bounded VC-dimension in general.

Let the projection of a set system $\mathcal{F} \subseteq 2^{[n]}$ to a set $X \subseteq [n]$ be

$$\mathcal{F}|_X = \{F \cap X \mid F \in \mathcal{F}\}.$$

Note that $X \in Sh(\mathcal{F})$ iff $\mathcal{F}|_X = 2^X$.

The main result of this section considers the projections of a set family from the perspective of extremality.

Theorem 5.1 *Let $\mathcal{F} \subseteq 2^{[n]}$ be family of VC-dimension $t \geq 1$ such that $\mathcal{F}|_X$ is s-extremal for all $2t + 1$ element subsets X of $[n]$. Let*

$$\mathcal{G} = \{H \subseteq [n] \mid H \cap X \in \mathcal{F}|_X \text{ for all sets } X \subseteq [n] \text{ of size } 2t + 1\}.$$

Then $\mathcal{G}$ contains $\mathcal{F}$, $\mathcal{G}$ is s-extremal and of VC-dimension t. Moreover, if $\mathcal{F}$ is s-extremal then we have $\mathcal{G} = \mathcal{F}$.

Before proving Theorem 5.1 we first present some observations about extremal set systems in general.

For a set system $\mathcal{F} \subseteq 2^{[n]}$ and $i \in [n]$ let $\mathcal{F}_0^{(i)}$ and $\mathcal{F}_1^{(i)}$ be as defined previously in Section 3. The downshift operation of a set family $\mathcal{F}$ by the element $i \in [n]$ is defined as follows:

$$
\begin{aligned}
D_i(\mathcal{F}) = &\{F \mid F \in \mathcal{F}, \ i \notin F\} \cup \{F \mid F \in \mathcal{F}, \ i \in F, \ F\backslash\{i\} \in \mathcal{F}\} \\
&\cup \{F\backslash\{i\} \mid F \in \mathcal{F}, \ i \in F, \ F\backslash\{i\} \notin \mathcal{F}\} \\
= &\{F\backslash\{i\} \mid F \in \mathcal{F}\} \cup \{F \mid F \in \mathcal{F}, \ i \in F, \ F\backslash\{i\} \in \mathcal{F}\}.
\end{aligned}
$$

It is not hard to see that D_i preserves s-extremality (e.g. [6], Lemma 1) and as already noted above, if $\mathcal{F}$ is s-extremal, then so is $\mathcal{F}_j^{(i)}$ for $i \in [n]$ and $j = 0, 1$.

For a set system $\mathcal{F} \subseteq 2^{[n]}$ and $i \in [n]$ we put

$$M_i(\mathcal{F}) = \mathcal{F}_0^{(i)} \cap \mathcal{F}_1^{(i)},$$
$$U_i(\mathcal{F}) = \mathcal{F}_0^{(i)} \cup \mathcal{F}_1^{(i)}.$$

Proposition 5.1 *Suppose that we are given a set system $\mathcal{F} \subseteq 2^{[n]}$ and an arbitrary element $i \in [n]$. Then if $\mathcal{F}$ is s-extremal then so are $M_i(\mathcal{F})$, $U_i(\mathcal{F})$ and $\mathcal{F}|_X$ for all $X \subseteq [n]$. Actually for $\mathcal{F}|_X$ we have that $Sh(\mathcal{F}|_X) = Sh(\mathcal{F})|_X$, more precisely a set $Y \subseteq X$ is in $Sh(\mathcal{F}|_X)$ iff it is in $Sh(\mathcal{F})$.*

Proof: The following equalities follow easily from the definitions:

$$M_i(\mathcal{F}) = \mathcal{F}_0^{(i)} \cap \mathcal{F}_1^{(i)} = D_i(\mathcal{F})_1^{(i)},$$
$$U_i(\mathcal{F}) = \mathcal{F}_0^{(i)} \cup \mathcal{F}_1^{(i)} = D_i(\mathcal{F})_0^{(i)}.$$

From these it follows that if $\mathcal{F}$ is s-extremal then so are $M_i(\mathcal{F})$ and $U_i(\mathcal{F})$, since we can obtain them from $\mathcal{F}$ using operations preserving s-extremality.

Next note that if $X = \{x_1, \ldots, x_m\}$ then $\mathcal{F}|_X$ is just $U_{x_1}(U_{x_2}(\ldots U_{x_m}(\mathcal{F})\ldots))$, thus if the original set system is s-extremal, then so is its projected version.

For the second part we only have to note that for some $Y \subseteq X$ we have that $Y \cap (F \cap X) = Y \cap F$ for all $F \subseteq [n]$, and the result follows. ∎

We say that a set $I \subseteq [n]$ is strongly traced or strongly shattered ([5], [6]) by a set system $\mathcal{F} \subseteq 2^{[n]}$ when there is a set $B \subseteq [n]\backslash I$ such that

$$B + 2^I = \{B \cup H \mid H \subseteq I\} \subseteq \mathcal{F}.$$

The collection of all sets strongly traced by $\mathcal{F}$ is denoted by $st(\mathcal{F})$. It can be shown that $|st(\mathcal{F})|$ is bounded from above by $|\mathcal{F}|$ (reverse Sauer inequality, [5], [6]), and a set system is called extremal with respect to the reverse Sauer inequality if $|st(\mathcal{F})| = |\mathcal{F}|$. The authors in [6] proved that a set system is extremal with respect to the original Sauer inequality exactly when it is extremal with respect to the reverse one, and thus in this case $Sh(\mathcal{F}) = st(\mathcal{F})$.

Similarly to the case of $Sh(\mathcal{F})$, $st(\mathcal{F})$ can also be obtained from the standard monomials of the vanishing ideal $I(\mathcal{F})$, namely, if viewed as a set of monomials, then it is the collection of those monomials which are standard monomials for all lexicographic term orders.

For a set system $\mathcal{F} \subseteq 2^{[n]}$ and a set $B \subseteq [n]$ denote the set family $\{I \subseteq [n]\backslash B \mid I + 2^B \subseteq \mathcal{F}\}$ by $\mathcal{F}(B)$. We remark that if $B = \{i_1, \ldots, i_m\} \subseteq [n]$, then $\mathcal{F}(B)$ is just $M_{i_1}(M_{i_2}(\ldots M_{i_m}(\mathcal{F})\ldots))$, and hence is s-extremal if $\mathcal{F}$ is

such. This together with the fact that s-extremality of a set system implies the connectedness of its inclusion graph proves in a simple way the 'if' direction of the following remarkable result of Bollobás and Radcliffe (Theorem 3 in [6]).

Theorem 5.2 *([6])* $\mathcal{F} \subseteq 2^{[n]}$ *is s-extremal iff* $G_{\mathcal{F}(B)}$ *is connected for every* $B \subseteq [n]$. ∎

Now we prepare the ground for the proof of Theorem 5.1 and return to the algebraic point of view. Let $\mathcal{F} \subseteq 2^{[n]}$ be an arbitrary family, and fix one term order $\prec$ on the monomials in $\mathbb{F}[x_1, \ldots, x_n]$. Suppose that we have at our disposal a Gröbner basis $\mathbb{G}$ of $I(\mathcal{F})$ with respect to a term order $\prec$ (e.g. in the s-extremal case we can compute one as described in Section 4). From $\mathbb{G}$ we can compute a Gröbner basis for $I(\mathcal{F}|_X)$: we only have to take the polynomials in $\mathbb{G}$ depending only on the variables x_i, $i \in X$. In general for a finite set of polynomials $\mathbb{G} \subseteq \mathbb{F}[x_1, \ldots, x_n]$ denote $\mathbb{G} \cap \mathbb{F}[x_i \mid i \in X]$ by $\mathbb{G}|_X$. The leading term $lt(f)$ of a nonzero polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$ with respect to $\prec$ is $lm(f)$ together with its coefficient from $\mathbb{F}$. The S-polynomial of nonzero polynomials f, g in $\mathbb{F}[x_1, \ldots, x_n]$ is

$$S(f, g) = \frac{L}{lt(f)} f - \frac{L}{lt(g)} g,$$

where L is the least common multiple of $lm(f)$ and $lm(g)$. Buchberger's theorem ([1] Theorem 1.7.4.) states that a finite set $\mathbb{G}$ of polynomials in $\mathbb{F}[x_1, \ldots, x_n]$ is a Gröbner basis for the ideal generated by $\mathbb{G}$ iff the S-polynomial of any two polynomials from $\mathbb{G}$ can be reduced to 0 using $\mathbb{G}$. (For more details on reduction and proofs see Chapter 1 of [1].)

Proof of Theorem 5.1: The facts $\mathcal{F} \subseteq \mathcal{G}$ and $VC - dim(\mathcal{G}) = t$ just follow immediately from the definition of $\mathcal{G}$.

Now fix one term order $\prec$ and one $2t + 1$ element subset X of $[n]$. Since by assumption $\mathcal{F}|_X$ is extremal, according to Proposition 4.1 the polynomials of the form $f_{S,H}$, where S is a minimal element outside $Sh(\mathcal{F}|_X)$ and H is the unique subset of S such that $H \notin (\mathcal{F}|_X)|_S$, together with $\{x_i^2 - x_i \mid i \in X\}$ form a Gröbner basis $\mathbb{G}_X$ of $I(\mathcal{F}|_X)$ with respect to $\prec$. We have

$$VC - dim(\mathcal{F}|_X) \leq VC - dim(\mathcal{F}) = t,$$

hence $|S| \leq t + 1$ for all polynomials of the form $f_{S,H}$ in $\mathbb{G}_X$. Write

$$\mathbb{G} := \bigcup_{X \subseteq [n], |X| = 2t+1} \mathbb{G}_X.$$

Note that a polynomial $f_{S,H}$ from $\mathbb{G}$ is a member of $\mathbb{G}_X$ for all $2t+1$ element subsets X for which $S \subseteq X$. First we prove using Buchberger's theorem that

$\mathbb{G}$ is a Gröbner basis of $\langle \mathbb{G} \rangle$ with respect to $\prec$. Take two polynomials $f, g \in \mathbb{G}$ and let m be the number of variables occurring in them. If $m \leq 2t + 1$ then there is some $2t + 1$ element set X such that $f, g \in \mathbb{G}_X$. However since $\mathbb{G}_X$ is a Gröbner basis of $I(\mathcal{F}|_X)$, $S(f, g)$ can be reduced to 0 using $\mathbb{G}_X$, and so using $\mathbb{G}$ with respect to $\prec$ as well. On the other hand $m > 2t + 1$ is possible only if $f = f_{S,H}$ and $g = f_{S',H'}$ for some appropriate sets $H \subseteq S, H' \subseteq S'$ such that $S \cap S' = \emptyset$ and $|S| = |S'| = t + 1$. The leading terms of $f_{S,H}$ and $f_{S',H'}$ are $\prod_{i \in S} x_i$ and $\prod_{j \in S'} x_j$ respectively, so we can write them in the following form:

$$f = f_{S,H} = \underbrace{\prod_{i \in S} x_i}_{\mathbf{x}_S} + f' \text{ and } g = f_{S',H'} = \underbrace{\prod_{j \in S'} x_j}_{\mathbf{x}_{S'}} + g',$$

where f' and g' depend on disjoint sets of variables.

$$S(f, g) = S(f_{S,H}, f_{S',H'}) = \mathbf{x}_{S'} f' - \mathbf{x}_S g'$$

Here if we replace $\mathbf{x}_{S'}$ by $-g'$ and $\mathbf{x}_S$ by $-f'$, the resulting polynomial will be identically 0, i.e. reducing $S(f, g)$ using $f, g \in \mathbb{G}$ gives 0. Moreover, the above reasoning works for all term orders $\prec$, so $\mathbb{G}$ is a Gröbner basis of $\langle \mathbb{G} \rangle$ for all term orders. Consider next

$$A := \mathbb{F}[x_1, \ldots, x_n] / \langle x_i^2 - x_i, i \in [n] \rangle.$$

Clearly we have $\{x_i^2 - x_i \mid i \in [n]\} \subseteq \mathbb{G}$ and so $\langle \mathbb{G} \rangle$ defines also an ideal of A. A is actually isomorphic to the ring of all functions from $\{0, 1\}^n$ to $\mathbb{F}$, which in turn is isomorphic to $\mathbb{F}^{2^n}$. In this ring every ideal is the intersection of maximal ideals and hence every ideal is a radical ideal. This implies that every ideal in A, in particular $\langle \mathbb{G} \rangle$ as well, is a vanishing ideal of some finite point set from $\{0, 1\}^n$. When considering the $0 - 1$ vectors as characteristic vectors, this finite point set also defines a finite set system. It is not difficult to see, that in case of $\langle \mathbb{G} \rangle$ the only possible candidate for this set system is $\mathcal{G}$ itself, so $\langle \mathbb{G} \rangle = I(\mathcal{G})$. However in this case we get that $\mathbb{G}$ is a Gröbner basis of $I(\mathcal{G})$ for all term orders and hence according to Proposition 4.1 $\mathcal{G}$ is s-extremal.

Finally we note that if $\mathcal{F}$ itself is already s-extremal, then according to Proposition 4.1 $\mathbb{G}$ is a Gröbner basis for $I(\mathcal{F})$ as well and so $\mathcal{F} = \mathcal{G}$. ∎

6 Concluding remarks

Concerning the structure of s-extremal set systems the question arises whether an extremal family can be built up from the empty system by adding sets to it one-by-one in such a way that at each step we have an s-extremal family.

Open problem 1 *For a nonempty s-extremal family $\mathcal{F} \subseteq 2^{[n]}$ does there always exist a set $F \in \mathcal{F}$ such that $\mathcal{F}\backslash\{F\}$ is still s-extremal?*

From Theorem 2 of [6] we know that $\mathcal{F}$ is s-extremal iff $2^{[n]}\backslash\mathcal{F}$ is s-extremal, thus the above question has an equivalent form:

Open problem 2 *For an s-extremal family $\mathcal{F} \subsetneq 2^{[n]}$ does there always exist a set $F \in \mathcal{F}$ such that $\mathcal{F} \cup \{F\}$ is still s-extremal?*

There are several special cases when the answer appears to be true:

1. If $\mathcal{F}$ is a nonempty down set ($F \in \mathcal{F}$ and $H \subseteq F$ then $H \in \mathcal{F}$), then $\mathcal{F}$ is extremal since $Sh(\mathcal{F}) = \mathcal{F}$. Moreover in this case if we omit a maximal element from $\mathcal{F}$ then it remains still a down set and so it will be still s-extremal.

2. If $\mathcal{F}$ is an extremal family of VC-dimension 1, then according to Proposition 2.1, if we omit a set corresponding to a leaf, i.e. to a vertex of degree 1 in $G_{\mathcal{F}}$, then the resulting set system will still be extremal.

3. Anstee in [3] constructed set systems $\mathcal{F} \subseteq 2^{[n]}$, $|\mathcal{F}| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2}$ without triangles, i.e. set systems with the property, that for all 3-element subsets F we have that $\mathcal{F}|_F$ does not contain all 2-element subsets of F. Note that in particular $VC - dim(\mathcal{F})$ is bounded from above by 2, hence we have that $Sh(\mathcal{F}) \subseteq \binom{[n]}{0} \cup \binom{[n]}{1} \cup \binom{[n]}{2}$, implying that $|Sh(\mathcal{F})| \leq \binom{n}{0} + \binom{n}{1} + \binom{n}{2}$. Comparing the sizes of $\mathcal{F}$ and $Sh(\mathcal{F})$ we obtain that such set systems are s-extremal.

Clearly any such set system $\mathcal{F}$ contains the extremal subsystem $\binom{[n]}{0} \cup \binom{[n]}{1}$. For the remaining part of these set systems Anstee's construction can be interpreted in an inductive way as follows:

- $\mathcal{F}_1 := \binom{[n]}{0} \cup \binom{[n]}{1}$

- For $k = 2, 3, \ldots, n$ suppose we already constructed $\mathcal{F}_{k-1}$. Let $\mathcal{G}_{k-1}$ be the collection of all $k - 1$-element sets in $\mathcal{F}_{k-1}$. Define G_{k-1} to be a graph, whose vertex set is $\mathcal{G}_{k-1}$ and there is an edge between $A, B \in \mathcal{G}_{k-1}$ exactly when $|A \triangle B| = 2$. Take a spanning tree T_{k-1} of G_{k-1}.

$$\mathcal{F}_k := \mathcal{F}_{k-1} \cup \{A \cup B \mid (A, B) \text{ is an edge of } T_{k-1}\}$$

- $\mathcal{F} := \mathcal{F}_n$

It is not hard to prove that when we add $A \cup B$, there will be a unique new element that gets into the family of shattered sets, namely $A \triangle B$, hence the resulting system after each step will be s-extremal. Reversing it, if $\mathcal{F}$ is such

an example, then its elements can be deleted one-by-one in such a way that the remaining set system is s-extremal after each step.

4. More generally one can consider set systems $\mathcal{F} \subseteq 2^{[n]}$ with the property, that for all t-element subsets F we have that $\mathcal{F}|_F$ does not contain all l-element subsets of F, for some l with $n \geq t \geq l \geq 0$. Füredi and Quinn in [13] constructed for all values $n \geq t \geq l \geq 0$ a set system $\mathcal{F}(n,t,l)$ with the desired property and of size $|\mathcal{F}(n,t,l)| = \sum_{i=0}^{t-1} \binom{n}{i}$. The same argument as above shows that $Sh(\mathcal{F}(n,t,l))$ consists of all sets of size at most $t-1$ and hence $\mathcal{F}(n,t,l)$ is s-extremal for all possible values. Their construction is as follows.

For $x_1, \dots, x_i \in [n]$, $x_1 < \cdots < x_i$ let

$$E(x_1, \dots, x_i) = \{x \in [n] \mid x = x_j \text{ for } j \leq l\}$$
$$\cup \{x \in [n] \mid x > x_l \text{ but } x \neq x_j \text{ for any } j > l\},$$

in particular $E(\emptyset) = \emptyset$. Let $\mathcal{F}(n,t,l)$ consist of all $E(x_1, \dots, x_i)$ where $i \leq t-1$. Order the sets of $\mathcal{F}(n,t,l)$ as follows: $E(X) \succ E(Y)$ if either $|X| > |Y|$, or $|X| = |Y|$ and $X \succ Y$ with respect to the standard lexicographic ordering. It is not hard to see, that if we remove the elements of $\mathcal{F}(n,t,l)$ with respect to this ordering one-by-one, starting from the largest one, then each time when we remove some $E(X)$, then X is eliminated from the family of shattered sets, hence after each step the resulting family will be still s-extremal.

Acknowledgments

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